1. A curve C is defined by the equations $z = x^2 + y^2$ and $y + z = 1$.

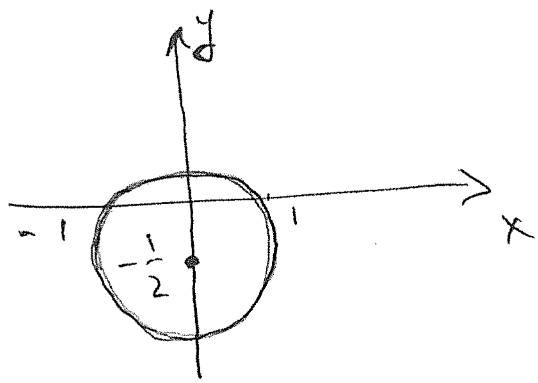
(a) (3 points) Find equations for the projection of C to the xy-plane and sketch the resulting curve.

$S = \{(x,y) \mid$ there is z corresponding to $(x,y)\}$

$z = 1-y$ and $1-y = x^2+y^2 \iff x^2+y^2+y = 1 \iff x^2 + (y+\frac{1}{2})^2 = \frac{5}{4}$

circle of radius $\frac{\sqrt{5}}{2}$.

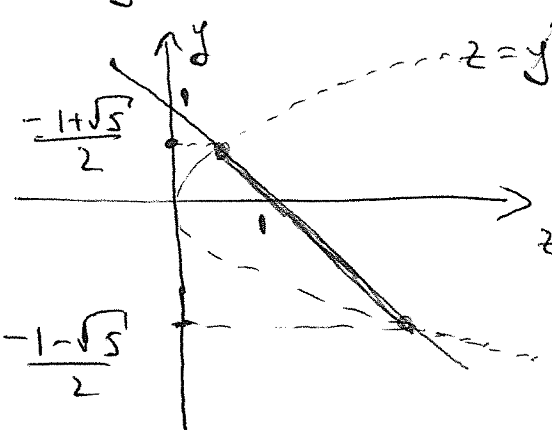
$\therefore S = \{(x,y) \mid x^2 + (y+\frac{1}{2})^2 = \frac{5}{4}\}$



(b) (4 points) Find equations for the projection of C to the yz-plane and sketch the resulting curve.

$S = \{(y,z) \mid$ there is x corresponding to $(y,z)\}$, i.e.,

$y+z = 1$ and $x^2 = z-y^2$, i.e., $z-y^2 \geq 0$



$z = 1-y$ and so

$z-y^2 \geq 0 \iff 1-y-y^2 \geq 0$

$\iff y^2+y-1 \leq 0$

$y^2+y-1 = 0 \iff y = \frac{-1 \pm \sqrt{5}}{2}$

$\therefore \frac{-1-\sqrt{5}}{2} \leq y \leq \frac{-1+\sqrt{5}}{2}$.

2. (a) (3 points) Find an equation defining the tangent plane to the surface $x^2 + 2y^2 = 4 - 3z^2$ at the point $(1, 0, 1)$.

$F(x,y,z) = x^2+2y^2+3z^2-4$ Surface: $F(x,y,z)=0$

$\nabla F = (2x, 4y, 6z)$ and $\nabla F(1,0,1) = (2,0,6) = 2(1,0,3)$

$\therefore \overline{n} = (1,0,3)$ is a normal vector to this surface at $(1,0,1)$.

Tangent plane: $\underline{1(x-1)+0\cdot(y-0)+3(z-1)=0}$

$\Leftrightarrow x-1+3z-3=0 \Leftrightarrow \underline{x+3z-4=0}$

(b) (2 points) Show that the tangent plane from part (a) does not intersect the line $x = 1 + 3t$, $y = 2$, $z = 2 - t$.

Line: $\overline{r}(t) = (1+3t, 2, 2-t) = (1,2,2) + t(3,0,-1)$, $t \in \mathbb{R}$

$\therefore \overline{v} = (3,0,-1)$ is $\parallel$ to this line

$\overline{v} \cdot \overline{n} = (3,0,-1)\cdot(1,0,3) = 3-3 = 0$

$\therefore \overline{v} \perp \overline{n}$

Also, $(1,2,2)$ does not satisfy $x^2+2y^2 = 4-3z^2$

$\therefore$ this point is not on the plane for (a) $\therefore$ Line and plane do not intersect.

(c) (2 points) Find the shortest distance between the line from part (b) and the plane from part (a).

Distance = dist($(1,2,2)$, plane $x+3z-4=0$) =

$$= \frac{|1+3\cdot 2-4|}{\sqrt{1+3^2}} = \underline{\frac{3}{\sqrt{10}}}.$$

3. (4 points) Find parametric equations for the curve C defined by

$$x^2 + 4y^2 = 4, \quad x + z = 1,$$

directed so that x is **increasing** when y is positive.

$$\begin{cases} x^2+4y^2=4 \\ x+z=1 \end{cases} \iff \begin{cases} \left(\frac{x}{2}\right)^2+y^2=1 \\ z=1-x \end{cases} \iff \begin{cases} \frac{x}{2}=\cos t \\ y=\sin t \\ z=1-x \end{cases}, \quad \text{for } 0 \le t \le 2\pi.$$

$y \ge 0$ if $\sin t \ge 0$, i.e., $0 \le t \le \pi$.
$x = 2\cos t$ is decreasing on $[0, \pi]$, i.e., we need to change direction/orientation of the curve.

$$\therefore \begin{cases} x = 2\cos(-t) \\ y = \sin(-t) \\ z = 1-x \end{cases}, \quad 0 \le t \le 2\pi \iff \boxed{\begin{cases} x = 2\cos t \\ y = -\sin t \\ z = 1-2\cos t \end{cases}, \; 0 \le t \le 2\pi}$$

4. (5 points) Suppose that equations $xyuv = 1$ and $x + y + u + v = 0$ define y and v as functions of x and u. Find $\dfrac{\partial y}{\partial x}$.

Method 1: ~~F(x, y, u, v)~~ $F(x,u,y,v) = xyuv - 1$

$G(x,u,y,v) = x+y+u+v$

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(y,v)}} \quad \therefore \frac{\partial y}{\partial x} = -\frac{\begin{vmatrix} yuv & xyu \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} xuv & xyu \\ 1 & 1 \end{vmatrix}} =$$

$$= -\frac{(yuv - xyu)}{(xuv - xyu)} = -\frac{yu(v-x)}{xu(v-y)} = \boxed{-\frac{y}{x}\cdot\frac{v-x}{v-y}}$$

(note: $u \neq 0$)

Method 2: $v = -(x+y+u) \quad \therefore \quad xyu(x+y+u) = -1$

$\Leftrightarrow \quad x^2yu + xuy^2 + xu^2y + 1 = 0$

Diff. wrt x (note: $y = y(x,u)$ and x and u are independent)

$$2xuy + x^2u\frac{\partial y}{\partial x} + uy^2 + 2xuy\frac{\partial y}{\partial x} + u^2y + xu^2\frac{\partial y}{\partial x} = 0$$

$$\therefore \frac{\partial y}{\partial x} = -\frac{2xuy + uy^2 + u^2y}{x^2u + 2xuy + xu^2} = -\frac{yu(2x+y+u)}{xu(x+2y+u)} =$$

$$= \boxed{-\frac{y}{x}\cdot\frac{2x+y+u}{x+2y+u}}$$

Remark: $\dfrac{\partial(F,G)}{\partial(y,v)} = xu(v-y)$ is 0 if $u=0$, and so y and v cannot be defined as f-s of x and u near such a point. Hence, we can assume that $u \neq 0$.

5. Find the limit or show that it does not exist:

(a) (4 points) $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$

$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \frac{\sin(xy)}{x^2+y^2} = \lim_{x\to 0} \frac{\sin 0}{x^2+0^2} = \lim_{x\to 0} 0 = 0$

$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=y}} \frac{\sin(xy)}{x^2+y^2} = \lim_{x\to 0} \frac{\sin(x^2)}{2x^2} = \frac{1}{2}\lim_{u\to 0} \frac{\sin u}{u} = \frac{1}{2}$

Since limits along $y=0$ and $y=x$ are not the same, we conclude that

$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$ DNE

(b) (2 points) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2-y^2)}{\cos(x^2+y^2)}$

$\lim_{(x,y)\to(0,0)} \sin(x^2-y^2) = \sin 0 = 0$

$\lim_{(x,y)\to(0,0)} \cos(x^2+y^2) = \cos 0 = 1$

$\therefore \lim_{(x,y)\to(0,0)} \frac{\sin(x^2-y^2)}{\cos(x^2+y^2)} = \frac{0}{1} = \boxed{0}.$

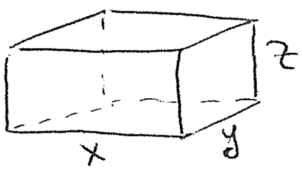
6. Find the dimensions of the rectangular box R with no top such that:

   1. the volume of R is $1\,m^3$,
   2. none of the sides of R is longer than $2\,m$,
   3. R has the least possible total surface area of its five faces.

Instructions:

(a) (4 points) Restate this problem as a minimization problem of a function $f = f(x, y)$ on some region D.

(b) (3 points) Solve this problem assuming that it is known that the values of f at all critical points inside D are smaller than its values on the boundary of D.

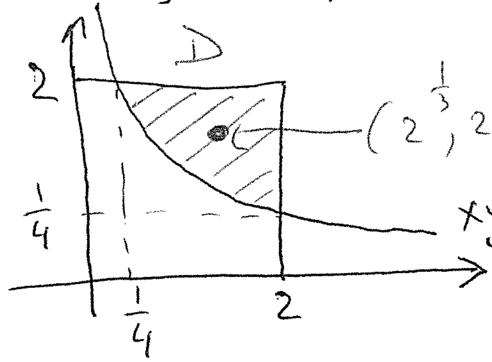


Volume $= xyz \; (= 1m^3)$, where $0 \le x, y, z \le 2$ (m)

Surface area $= xy + 2xz + 2yz$ is minimal.

$\therefore z = \frac{1}{xy}$, $0 \le x, y \le 2$ and $\frac{1}{xy} \le 2 \Leftrightarrow xy \ge \frac{1}{2}$

Hence, surface area is $f(x,y) = xy + \frac{2}{x} + \frac{2}{y}$, $x, y \le 2$ and $xy \ge \frac{1}{2}$



a) We need to find the abs. minimum value of $f(x,y) = xy + \frac{2}{x} + \frac{2}{y}$ on the region $D = \{(x,y) \mid x \le 2, y \le 2, xy \ge \frac{1}{2}\}$.

b) $\frac{\partial f}{\partial x} = y - \frac{2}{x^2}$, $\frac{\partial f}{\partial y} = x - \frac{2}{y^2}$ and

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Leftrightarrow \begin{cases} y = \frac{2}{x^2} \\ x = \frac{2}{y^2} \end{cases} \Leftrightarrow \begin{cases} y = \frac{2}{x^2} \\ x = \frac{2x^4}{4} \end{cases} \Leftrightarrow \begin{cases} y = \frac{2}{x^2} \\ x^3 = 2 \end{cases}$$

$\Leftrightarrow x = y = 2^{\frac{1}{3}}$.

$\therefore$ Critical point of f is $(2^{\frac{1}{3}}, 2^{\frac{1}{3}})$ (it is inside D).

$f(2^{\frac{1}{3}}, 2^{\frac{1}{3}}) = 3 \cdot 2^{2/3}$.

$\therefore$ Abs. minimum value is $3 \cdot 2^{2/3}$ (attained when $x = y = 2^{\frac{1}{3}}$ (m) and $z = 2^{-\frac{2}{3}}$ (m)).